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UNSTEADY FREE CONVECTION HEAT TRANSFER

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ABSTRACT

An analysis is made for laminar free-convection heat transfer from a vertical surface whose temperature varies with time in an arbitrary manner. A quantitative criterion is derived which gives the conditions under which unsteady heat-transfer calculations can be carried out simply by using heat-transfer coefficients for the quasi-steady state (instantaneous steady state). Results are given for Prandtl numbers ranging from 0.03 to 10. The findings reported here should also serve to define quasi-steady conditions for turbulent free-convection heat transfer.

NOMENCLATURE

c_p	specific heat at constant pressure
F	velocity function for quasi-steady state
f_o, f_1	functions of η
g	acceleration of gravity
h	local heat-transfer coefficient, $q/\Delta T$
k	thermal conductivity
L	plate height
Pr	Prandtl number, $c_p\mu/k$
Q	over-all heat-transfer rate
q	local heat-transfer rate per unit area
T	static temperature
ΔT	temperature difference, $\lvert T_w - T_\infty \rvert$
$\dot{\Delta T}, \ddot{\Delta T}$	time derivatives, $d/dt(\Delta T)$, $d^2/dt^2(\Delta T)$
t	time
u	velocity component in x-direction
v	velocity component in y-direction

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x	coordinate measuring distance along plate from leading edge
y	coordinate measuring distance normal to plate
Y	transformed coordinate, $\int_0^y (\rho/\rho_\infty)dy$
β	coefficient of thermal expansion, $-\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_p$
η	similarity variable, $[g\beta(\Delta T)/4\nu^2]^{1/4}Y/x^{1/4}$
Θ	temperature variable for quasi-steady state
θ	dimensionless temperature, $(T - T_\infty)/(T_w - T_\infty)$
θ_o,θ_1	functions of η
λ_o,λ_1	series expansion parameters defined by eq. (8)
μ	absolute viscosity
ν	kinematic viscosity
ρ	density
ψ	stream function

Subscripts:

inst	instantaneous
qs	quasi-steady
w	wall
∞	ambient

INTRODUCTION

In a number of important technical applications, it is necessary to compute the free-convection heat transfer from a surface whose temperature changes with time. This very difficult problem is appreciably simplified when it is supposed that at each and every moment, there exists an instantaneous steady state. Under such an assumption, the steady state relationships for the heat-transfer coefficient are used in conjunction with the instantaneous temperature difference to compute a heat transfer rate. It is customary to apply the term quasi-steady to describe this situation where the transient passes through a sequence of instantaneous steady states.

In reality, there is always a difference between the actual instantaneous heat transfer and the quasi-steady value. The extent of the deviation depends both on the rapidity of the changes in surface temperature and on the response characteristics of the fluid. Clearly, quasi-steady heat transfer would not be expected if the changes in surface temperature are exceedingly rapid.

Because of the tremendous practical simplifications associated with the quasi-steady assumption, it is important to know the conditions under which it can be invoked with negligible error. The aim of this analysis is to determine a quantitative criterion to distinguish when the heat transfer is essentially quasi-steady. The system chosen for study is a vertical plate as pictured in Figure 1. The surface temperature T_w is spatially uniform, but is permitted to take on arbitrary, but continuously differentiable, variations with time. The ambient temperature T_∞ is taken as constant.

Results are presented here for Prandtl numbers ranging from 0.03 to 10. The lower end of this range corresponds to liquid metals; the span from 0.65 to 1.1 is associated with gases; while liquids such as water, organic solvents, and inorganic salts have Prandtl numbers between 1 and 10.

Previous analytical work on unsteady free convection has been rather sparce. Using the Karman-Pohlhausen technique, Siegel (1) studied the special problem of a sudden step change in surface temperature or heat flux. Illingworth (2) and Sugawara and Michiyoshi (3) have also analyzed situations involving step changes in surface temperature.

Readers who are primarily interested in results are invited to pass over the section on ANALYSIS.

ANALYSIS

Governing equations and boundary conditions. - To achieve our ultimate goal of finding a criterion for the existance of quasi-steady conditions, we must first analyze the velocity and temperature distributions in the boundary layer adjacent to the vertical plate. The problem is, of course, governed by the basic conservation laws: mass, momentum, and energy; and it is these which are the starting point for our study. The mathematical expression of these laws appropriate to unsteady flow in a boundary layer on a vertical plate is

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \tag{1}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \pm g\beta\rho(T - T_\infty) + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{2}$$

$$\rho c_p\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) \tag{3}$$

where $\beta = 1/T_\infty$ for gases. The essential term in these equations is $g\beta\rho(T - T_\infty)$, which represents the buoyancy force responsible for the free convection motion. Viscous dissipation and work against the gravity field have been neglected in accordance with the usual practice in free convection. By affixing the plus-minus signs to the buoyancy term of equation (2), we are able to simultaneously analyze both of the physical situations pictured in figure 1. The left hand sketch (a), associated with the plus sign, depicts the case where the wall temperature exceeds ambient and the boundary layer flow is upward. The right hand sketch (b), with which the minus sign is used, represents the case where $T_w < T_\infty$ and the flow is downward in the boundary layer. Throughout the analysis and the presentation of results, there will be no need to make any particular distinction between these two situations.

The statement of the problem is completed by giving the boundary conditions as follows:

$$\left.\begin{aligned} u &= 0 \\ v &= 0 \\ T &= T_w(t) \end{aligned}\right\} y = 0 \qquad \qquad \left.\begin{aligned} u &\to 0 \\ T &\to T_\infty \end{aligned}\right\} y \to \infty \tag{4}$$

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where $T_w(t)$ is any arbitrary, but continuously differentiable function of time. The requirement that each velocity component vanishes at the plate surface arises from the no-slip condition of viscous flow ($u = 0$) and the impermeability of the wall to mass ($v = 0$).

Proceeding in a general way, we observe that the conservation of mass equation (1) may be satisfied by defining a stream function ψ by the relations (ref. 4, eq. (5))

$$u = \frac{\rho_\infty}{\rho} \frac{\partial \psi}{\partial y}, \qquad v = - \frac{\rho_\infty}{\rho}\left(\frac{\partial \psi}{\partial x} + \frac{\partial}{\partial t} \int_0^y \frac{\rho}{\rho_\infty} dy\right) \tag{1a}$$

Then, by replacing u and v in favor of ψ and introducing the following new variables

$$\theta = \frac{T - T_\infty}{T_w - T_\infty}, \qquad Y = \int_0^y \frac{\rho}{\rho_\infty} dy \tag{5}$$

equations (2) and (3) may be rephrased as

$$\frac{\partial^2 \psi}{\partial Y \partial t} + \frac{\partial \psi}{\partial Y} \frac{\partial^2 \psi}{\partial Y \partial x} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial Y^2} = g\beta(\Delta T)\theta + \nu \frac{\partial^3 \psi}{\partial Y^3} \tag{2a}$$

$$\frac{\partial \theta}{\partial t} + \theta \frac{\dot{\Delta T}}{\Delta T} + \frac{\partial \psi}{\partial Y} \frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \theta}{\partial Y} = \frac{\nu}{Pr} \frac{\partial^2 \theta}{\partial Y^2} \tag{3a}$$

where $\Delta T = |T_w - T_\infty|$, $\qquad \dot{\Delta T} = d(\Delta T)/dt$

To simplify the treatment, we have neglected the variation of fluid properties in liquids. In the case of gases, it has been supposed that the properties may be approximated by $\rho\mu$ = constant, ρk = constant, c_p = constant, and as a consequence, the kinematic viscosity in equations (2a) and (3a) is evaluated at ambient conditions.

In terms of these new variables, the boundary conditions become

$$\left.\begin{aligned} \partial\psi/\partial Y &= 0 \\ \partial\psi/\partial x &= 0 \\ \theta &= 1 \end{aligned}\right\} Y = 0 \qquad\qquad \left.\begin{aligned} \psi_Y &\rightarrow 0 \\ \theta &\rightarrow 0 \end{aligned}\right\} Y \rightarrow \infty \tag{4a}$$

Having completed the formulation, we now turn to the task of solving the formidable mathematical problem represented by the simultaneous partial differential equations (2a) and (3a) for ψ and θ as functions of x, Y, and t.

SOLUTIONS

As has already been mentioned, the aim of this analysis is to investigate the conditions under which the actual instantaneous heat transfer deviates only slightly from the quasi-steady value. With this in mind, it is natural to seek a solution for the temperature and velocity distributions in the form of a series expansion about the quasi-steady state.

As a prelude to the series, we recall that for steady state conditions, the stream function ψ and dimensionless temperature θ are written as

$$\psi = [64g\beta(\Delta T)\nu^2x^3]^{1/4}F(\eta), \qquad \theta = \Theta(\eta) \tag{6a}$$

where

$$\eta = [g\beta(\Delta T)/4\nu^2]^{1/4}Y/x^{1/4} \tag{6b}$$

The variable η is the well-known similarity parameter first used by Schmidt and Beckmann.

Then, shifting our attention to the unsteady situation, we expand ψ and θ in series about the quasi-steady state (i.e., the instantaneous steady state). So,

$$\psi = [64g\beta(\Delta T)\nu^2x^3]\left\{F(\eta) + \lambda_o f_o(\eta) + \lambda_1 f_1(\eta) + . . .\right\} \tag{7a}$$

$$\theta = \Theta(\eta) + \lambda_o\theta_o(\eta) + \lambda_1\theta_1(\eta) + \ldots \tag{7b}$$

where

$$\lambda_o = \frac{\dot{\Delta T}}{\Delta T}\left[\frac{x}{g\beta(\Delta T)}\right]^{1/2}, \qquad \lambda_1 = \frac{\ddot{\Delta T}}{\Delta T}\left[\frac{x}{g\beta(\Delta T)}\right], \ldots \tag{8}$$

The expansion parameters $\lambda_o, \lambda_1, \ldots$ may be derived either by dimensional arguments or by mathematical analysis. As shown in the Appendix, these parameters have a clear physical meaning which may be represented by the following ratio

$$\frac{\text{time for diffusion of temp. changes across bl}}{\text{time characteristic of wall temp. changes}} \tag{9}$$

where bl is an abbreviation for boundary layer. Clearly, this ratio will be small when the boundary layer responds promptly to impressed changes in wall temperature. For quickly responding boundary layers, i.e., small values of $\lambda_o, \lambda_1, \ldots$, equations (7a) and (b) are consistent with the intuitive feeling that the velocity and temperature distributions are quasi-steady.

In the development which follows, it will be supposed that λ_o is substantially larger than $\lambda_o, \lambda_1, \ldots$, and as a consequence, the series (7a) and (b) are truncated after the second term. In practical terms, this assumption may be met when $\dot{\Delta T}$ is much larger than the higher derivatives. It is important to note that the retention of additional terms of the series would not alter the analysis, only the amount of numerical computations would be substantially increased.

Having settled on a series form for the solutions, we now return to the differential equations (2a) and (3a). Introducing the expansions (7a) and (b) for ψ and θ and grouping terms according to whether they are multiplied by $\lambda_o, \lambda_1, \ldots$, we are led to the following set of ordinary differential equations:

$$\Theta'' + 3(Pr)F\Theta' = 0 \qquad \Theta(0) = 1, \qquad \Theta(\infty) = 0 \tag{10a}$$

$$F''' + 3FF'' - 2(F')^2 + \Theta = 0 \qquad F(0) = F'(0) = 0, \qquad F'(\infty) = 0 \tag{10b}$$

$$\theta_0'' + Pr(3F\theta_0' - 2F'\theta_0) - Pr(2\Theta + 0.5\eta\Theta' - 5f_0\Theta') = 0$$
$$\theta_0(0) = \theta_0(\infty) = 0 \tag{11a}$$

$$f_0''' + 3Ff_0'' - 6F'f_0' + 5F''f_0 - (F' + 0.5\eta F'' - \theta_0) = 0$$
$$f_0(\theta) = f_0'(0) = f_0'(\infty) = 0 \tag{11b}$$

The boundary conditions have been evaluated by introducing the series for ψ and θ into equation (4a).

Equations (10a) and (b) may be recognized as coinciding with the governing equations for the steady-state free-convection problem, although here they apply to the quasi-steady situation. Numerical solutions* of this pair of simultaneous, nonlinear, ordinary differential equations has been carried out on an IBM 650 digital computer for Prandtl numbers of 0.03, 0.72, and 10. Then, using these solutions as input data, equations (11a) and (b) were also solved for the same Prandtl numbers.

The important numerical results which are used in the heat transfer calculations are listed in table I.

TABLE I. - FUNCTIONS NEEDED IN HEAT TRANSFER COMPUTATIONS

Pr	$-\Theta'(0)$	$-\theta_0'(0)$	$\theta_0'(0)/\Theta'(0)$
0.03	0.1346	0.1317	0.9785
.72	.5046	.7721	1.530
10	1.169	4.605	3.938

HEAT TRANSFER RESULTS

Local and over-all heat transfer. - The instantaneous local heat flux at the plate surface q_{inst} may be calculated by application of Fourier's law

*The numerical integration technique is outlined in reference 5.

$$q = -\left(k\,\frac{\partial T}{\partial y}\right)_{y=0} = -\Delta T\left(k\,\frac{\partial \theta}{\partial y}\right)_{y=0} \tag{12}$$

After introducing the series expansion (7b) for θ and taking account of the definition (6b) for η, the expression for q becomes

$$q_{inst} = -\frac{k\Delta T}{x^{1/4}}\left[\frac{g\beta(\Delta T)}{4\nu^2}\right]^{1/4}[\Theta'(0) + \lambda_0\theta_0'(0) + \ldots] \tag{13}$$

where $\Theta'(0)$, $\theta_0'(0)$, ..., are abbreviations for $(d\Theta/d\eta)_{\eta=0}$, ...

The quasi-steady heat transfer q_{qs} is given by

$$q_{qs} = -\frac{k\Delta T}{x^{1/4}}\left[\frac{g\beta(\Delta T)}{4\nu^2}\right]^{1/4}\Theta'(0) \tag{14}$$

Then, the important relationship between the instantaneous and the quasi-steady heat transfer is found by combining equations (13) and (14),

$$\frac{q_{inst}}{q_{qs}} = 1 + \frac{\dot{\Delta T}}{\Delta T}\left[\frac{x}{g\beta(\Delta T)}\right]^{1/2}\frac{\theta_0'(0)}{\Theta'(0)} + \ldots \tag{15}$$

where we have evaluated λ_0 from the defining equation (8).

From table I, it is seen that $\theta_0'(0)/\Theta'(0)$ is positive for all Prandtl numbers. Further, ΔT is always positive; being evaluated as $T_W - T_\infty$ when the wall temperature exceeds ambient and as $T_\infty - T_W$ when the wall is cooler than ambient. So, it may be concluded from equation (15) that the instantaneous heat transfer exceeds the quasi-steady value when the temperature difference $|T_W - T_\infty|$ increases with time ($\dot{\Delta T} > 0$) and is less than quasi-steady when the temperature difference is decreasing. A further conclusion may be drawn from the fact that the ratio $\theta_0'(0)/\Theta'(0)$ decreases steadily with Prandtl number; from which it follows from equation (15) that low Prandtl number fluids are less likely to experience deviations from quasi-steady heat transfer than are high Prandtl number fluids. This trend might have been intuitively anticipated as a consequence of the relatively high thermal diffusivity of liquid

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metals. The proceeding remarks apply with full certainty only when the neglected terms of the series are very small.

An alternate form of the results may be obtained by introducing heat transfer coefficients as follows:

$$h_{inst} = \frac{q_{inst}}{\Delta T}, \qquad h_{qs} = \frac{q_{qs}}{\Delta T}$$

Then, equation (15) may be rephrased as

$$\frac{h_{inst}}{h_{qs}} = 1 + \frac{\dot{\Delta T}}{\Delta T}\left[\frac{x}{g\beta(\Delta T)}\right]^{1/2} \frac{\theta_0'(0)}{\Theta'(0)} + \ldots \tag{15a}$$

Equation (15a) is in a form useful for the interpretation of heat transfer coefficients obtained under transient conditions.

Next, we compute the heat transfer Q from the entire surface of the plate by integrating the local heat transfer, i.e.,

$$Q = \int_0^L q\ dx \tag{16}$$

where the plate width has been chosen as unity. The integral is carried out successively for the instantaneous and quasi-steady situations using q from equations (13) and (14) respectively, and the results may be represented as the following ratio

$$\frac{Q_{inst}}{Q_{qs}} = 1 + \frac{1}{2}\frac{\dot{\Delta T}}{\Delta T}\left[\frac{L}{g\beta(\Delta T)}\right]^{1/2} \frac{\theta_0'(0)}{\Theta'(0)} + \ldots \tag{17}$$

Comparison of this expression with equation (15) shows that the over-all heat transfer deviates less from quasi-steady then does the local heat transfer at $x = L$. The finding is made plausible by noting that the over-all heat transfer includes contributions from upstream locations (near $x = 0$) where the boundary layer is thinner and more responsive to impressed changes.

<u>Criterion for quasi-steady heat transfer</u>. - We can now proceed to find a criterion for distinguishing the conditions under which the quasi-steady

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relationship may be used with sufficient accuracy in unsteady heat transfer calculations. Suppose that an accuracy of 5 percent is sufficient for the local heat transfer for many applications. Then, from equation (15) in conjunction with table I, we are able to find the values of $\frac{\dot{\Delta T}}{\Delta T}\left[\frac{x}{g\beta(\Delta T)}\right]^{1/2}$ which lead to 5 percent deviations of q_{inst} from q_{qs}. The results thus obtained are plotted as the upper curve of figure 2. Clearly, for values of $\frac{\dot{\Delta T}}{\Delta T}\left[\frac{x}{g\beta(\Delta T)}\right]^{1/2}$ lower than those given by this curve, the deviation of q_{inst} from q_{qs} is less than 5 percent.

Alternately, if an accuracy of 2 percent is sufficient for the local heat transfer calculation, we can compute a second group of values for $\frac{\dot{\Delta T}}{\Delta T}\left[\frac{x}{g\beta(\Delta T)}\right]^{1/2}$ to represent the criterion for quasi-steady heat transfer. These results are shown as the lower curve of figure 2.

So, to determine whether the heat transfer for a particular experiment is quasi-steady, we would check whether the values of $\frac{\dot{\Delta T}}{\Delta T}\left[\frac{x}{g\beta(\Delta T)}\right]^{1/2}$ occurring in that experiment fall below the selected curve of figure 2. Alternately, for design purposes, figure 2 immediately gives the values of $\frac{\dot{\Delta T}}{\Delta T}\left[\frac{x}{g\beta(\Delta T)}\right]^{1/2}$ which permit application of the quasi-steady heat transfer relationships.

APPLICATION TO ECKERT-SOEHNGEN EXPERIMENTS

An experimental determination of free convection heat transfer coefficients on a vertical plate has been carried out by Eckert and Soehngen (ref. 6) using the transient technique. Their apparatus consisted of a copper plate which was preheated in a furnace and then suspended vertically in air to cool by free convection and radiation. It was assumed that quasi-steady conditions existed, so that the measurements were reported as steady state results. Values of $\frac{\dot{\Delta T}}{\Delta T}\left[\frac{x}{g\beta(\Delta T)}\right]^{1/2}$ from Eckert and Soehngen's experiments have been plotted as a vertical band on the lower part of the figure. It is seen that the conditions of the experiment fall well within both the 2 and 5 percent criteria for quasi-steady heat transfer, thereby verifying the supposition of the experimenters.

CONCLUDING REMARKS

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Stability questions. - It is natural to inquire as to the effect of the unsteady wall temperature on the stability of the free convection flow. Since an adequate stability analysis of the steady free-convection boundary layer has yet to be given, it would appear that such an analysis for the unsteady situation is not now within range. So, we must confine ourselved to intuitive conjectures about the role of the time-dependent wall temperature.

Our interest here has been in flows which are almost quasi-steady; that is, in flows where the boundary layer is able to follow the changes in wall temperature quite closely. At first thought, it might be expected that for these conditions, the stability of the unsteady flow would not be too different from that of the steady flow. In particular, the expectation seems quite reasonable for flows where the temperature difference $|T_w - T_\infty|$ is steadily increasing. However, when $|T_w - T_\infty|$ is steadily decreasing, it would seem that the flow would be somewhat more prone to instability; while a flow in which $|T_w - T_\infty|$ alternately increased and decreased would likely be even less stable.

Application to turbulent flows. - Although the analysis given here is for laminar flow conditions, the findings may have a wider utility. Since the response of a turbulent flow should be more rapid than that of laminar flow, the criteria for quasi-steady heat transfer as given on figure 2 should certainly also serve for turbulent flow.

APPENDIX

PHYSICAL INTERPRETATION OF $\lambda_o, \lambda_1, \ldots$

We want to show that the expansion parameters $\lambda_o, \lambda_1, \ldots$ given by equation (8) are related to the ratio of the time for diffusion across the boundary layer to the time characterising the impressed wall temperature changes.

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From one-dimensional diffusion theory, it may be recalled that the time required for changes to diffuse across a layer of thickness δ is proportional to

$$\delta^2/\nu \qquad (18)$$

On the other hand, the boundary layer thickness for free convection is given by

$$\frac{\delta}{x} \sim Gr_x^{-1/4} = \left(\frac{g\beta(\Delta T)x^3}{\nu^2}\right)^{-1/4} \quad \text{or} \quad \delta \sim \left(\frac{\nu^2 x}{g\beta(\Delta T)}\right)^{1/4} \qquad (19)$$

where Gr_x is the Grashof number. Introducing this into equation (18), the diffusion time is found to be proportional to

$$\left(\frac{x}{g\beta(\Delta T)}\right)^{1/2} \qquad (18a)$$

The time characterising the changes in wall temperature cannot be given by a single quantity, but rather, will be represented by the following sets of times

$$\frac{\Delta T}{\dot{\Delta T}}, \quad \left(\frac{\Delta T}{\ddot{\Delta T}}\right)^{1/2}, \quad \left(\frac{\Delta T}{\dddot{\Delta T}}\right)^{1/3}, \ldots \qquad (20)$$

After forming the quotient of equation (18a) with the successive members of equation (20), we see that $\lambda_o, \lambda_1, \ldots$ are indeed related to these ratios.

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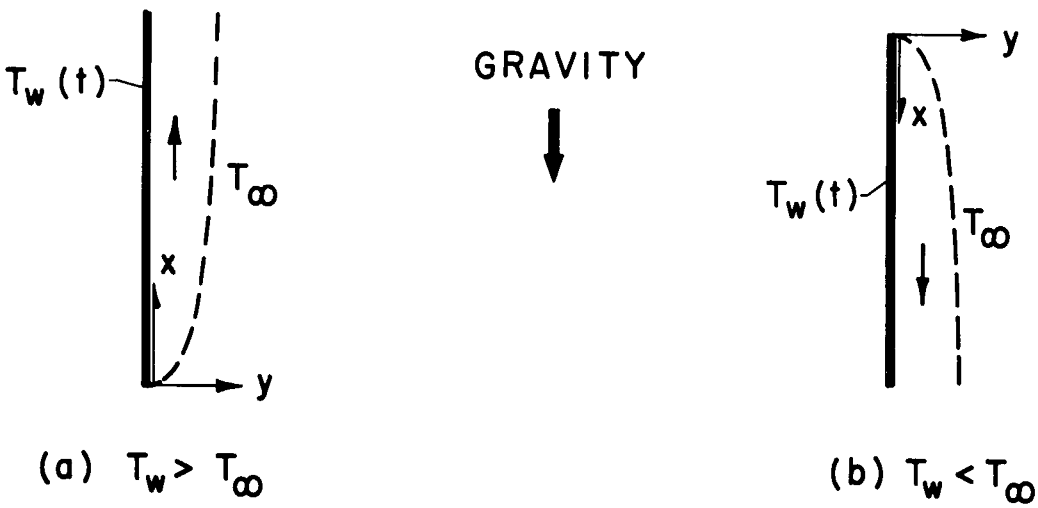

Fig. 1. - Physical model and coordinates.

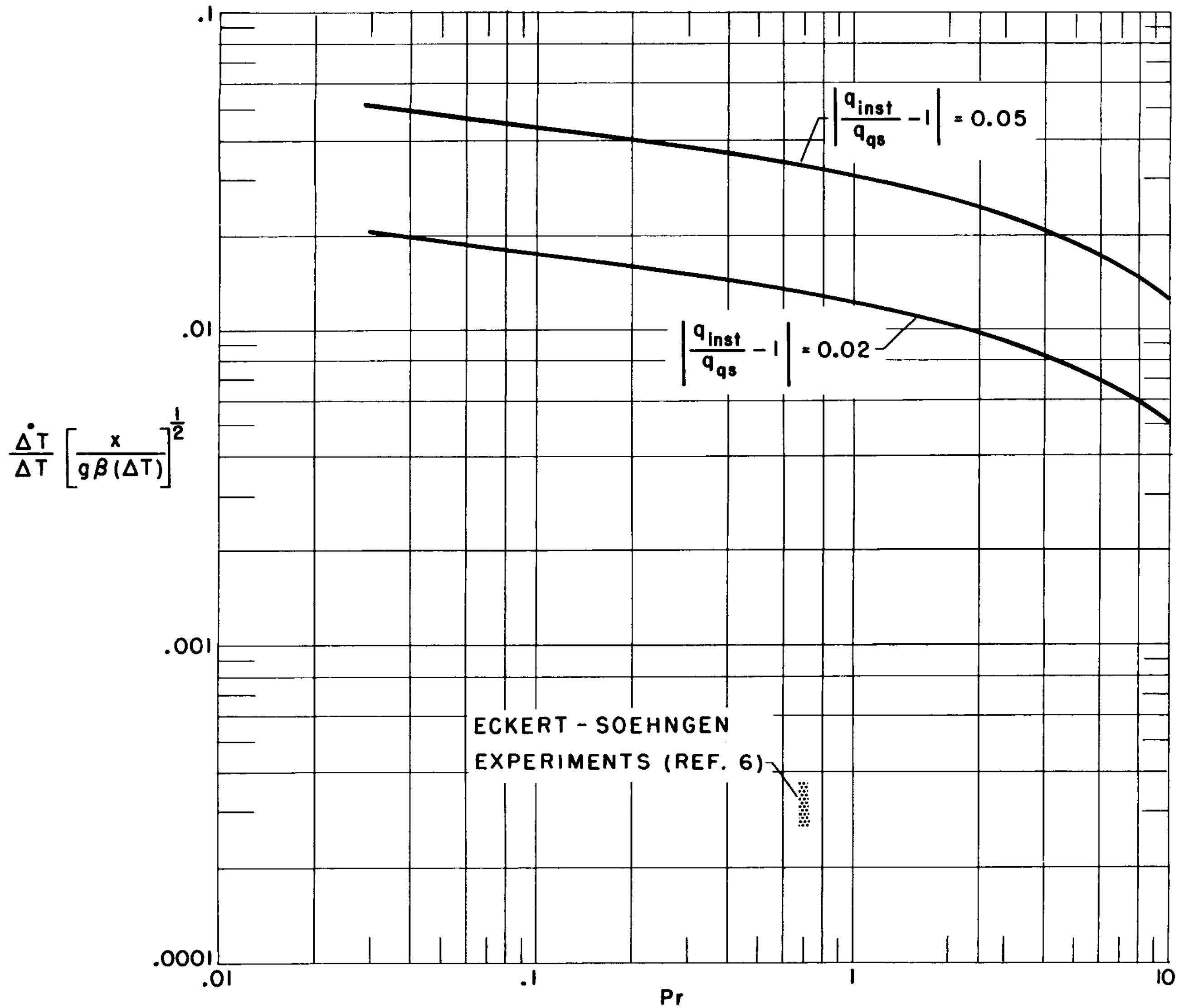

Fig. 2. - Criterion for 2% and 5% deviations of q_{inst} from q_{qs}.